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Open problems

Andrzej Schinzel

Is it true that for

$$n \equiv 3, 4, 7, 12, 15 \text{ or } 19 \mod 24$$

with $n > 268$ there exist integers x_1, x_2, x_3, x_4 with $(x_i, x_j) = 1$ for all $1 \leq i < j \leq 4$ such that

$$n = x_1^2 + x_2^2 + x_3^2 + x_4^2\,?$$

(For $n = 100$ and $n = 268$ such integers do not exist.)

Andrzej Schinzel (W. Bednarek)

For what values of n there exist monic polynomials $f \in \mathbb{Z}[x]$ of degree n such that the polynomial $f(x)^2 - 1$ has $2n$ integer roots, counting the multiplicities? (For $n = 1$ and $n = 2$ such polynomial exist.)

Kálmán Győry

Let $a \neq 0$ be an integer and let $p_1, \ldots, p_s$ be primes. Denote $P = \max\{p_1, \ldots, p_s\}$. Consider the equation

$$x - y = a$$

where x, y are integers composed only of $p_1, \ldots, p_s$. There is a bound for the unknowns

$$|x|, |y| \leq C_1 |a|^{C_2}$$

where $C_i = C_i(P, s)$, $i = 1, 2$, are effective constants.
Does there exist a bound for $|x|$ and $|y|$ such that C_2 depends only on s?

Attila Pethő

Let $a_1, \dots, a_n \in \mathbb{Z} \setminus \{0\}$ and let $b_1, \dots, b_n \geq 2$ be integers. Consider the polynomial

$$\Big(\cdots\big((x+a_1)^{b_1}+a_2\big)^{b_2}+\cdots+a_n\Big)^{b_n} = x^m + p_{m-1}x^{m-1} + \cdots p_0 \in \mathbb{Z}[x]$$

where $m = b_1 \cdots b_n$. Does there exist a constant $c > 0$ such that

$$\big|\{i : p_i \neq 0\}\big| \geq c\, b_1 \cdots b_n\,.$$

Attila Pethő

Let n be a positive integer, $\mathcal{C}_n = [0,1]^n$ and

$$\Delta = \Delta(t_1, \dots, t_n) = \prod_{1 \leq j < k \leq n} (t_j - t_k).$$

In 1944 A. Selberg proved the beautiful formula

$$\begin{aligned} S_n(\alpha, \beta, \gamma) &= \int_{\mathcal{C}_n} \prod_{j=1}^{n} t_j^{\alpha-1}(1-t_j)^{\beta-1}|\Delta|^{2\gamma} dt_1 \dots dt_n \\ &= \prod_{j=0}^{n-1} \frac{\Gamma(\alpha+j\gamma)\Gamma(\beta+j\gamma)\Gamma(1+(j+1)\gamma)}{\Gamma(\alpha+\beta+(n+j-1)\gamma)\Gamma(1+\gamma)}, \end{aligned}$$

which is valid for complex parameters α, β, γ such that $\Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > -\min\{1/n, \Re(\alpha)/(n-1), \Re(\beta)/(n-1)\}$.
For which values of α, β, γ is $S_n(\alpha, \beta, \gamma) \in \mathbb{Q}$ for every n and for which values of α, β, γ is $S_n(\alpha, \beta, \gamma) \in \overline{\mathbb{Q}}$ for every n?
For example I proved with S. Akiyama that $S_n(1, 1, 1/2)$ is the reciprocal of an integer.

Zuzana Masáková – About s-convex sets

We have a binary operation on $\mathbb{R}$ defined by $x \oplus y := sx + (1-s)y$ with a fixed real parameter s. Denote $D(s)$ the closure of $\{0,1\}$ under the operation $\oplus$. We say that a set $\Lambda \subset \mathbb{R}$ is *uniformly discrete*, if there exists $r > 0$ such that $|x-y| \geq r$ for every $x, y \in \Lambda$,

$x \neq y$. Λ is *relatively dense*, if there exists $R < +\infty$, such that every interval of length R contains an element of Λ.
It is known [2] that uniform discreteness of $D(s)$ implies that s is an algebraic integer. If $s \notin (0,1)$ is a totally real algebraic integer having all conjugates in $(0,1)$, then $D(s)$ is uniformly discrete. For a quadratic integer s, this is an equivalence.
It has been shown [1] that for s or $1-s \in \{-\frac{1}{2}(1+\sqrt{5}), -(1+\sqrt{2}), 2+\sqrt{3}\}$, the set $D(s)$ is also relatively dense, since in this case

$$D(s) = \{x \in \mathbb{Z}[s] : x' \in [0,1]\}\,, \tag{1}$$

where x' is the Galois image of x in the field $\mathbb{Q}(s)$. In particular, the distances between consecutive points of $D(s)$ take only three values, from which the distance 1 is reached only once (between 0 and 1).
Although (1) is not true for other quadratic integers $s \notin (0,1)$ with conjugate in $(0,1)$, we conjecture that $D(s)$ is still a relatively dense set. Prove.

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References

[1] Z. Masáková, J. Patera, E. Pelantová, *Exceptional algebraic properties of the three quadratic irrationalities observed in quasicrystals* Canad. J. Phys. **79**, (2001), 687–696.

[2] R. G. E. Pinch, *a-convexity,* Math. Proc. Camb. Phil. Soc. **97**, (1985), 63–68

Eliza Wajch

It is unprovable in ZF that for every uncountable set S the group $\mathbb{Z}_2^S$ (with the discrete topology) is not first-countable. Is it unprovable in ZF that $\mathbb{R}^S$ (with the usual topology) is not first-countable for uncountable S?