Representing integers as sums or differences of general power products

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Problems concerning representations of integers as linear combinations of power products has a large literature (see e.g. the corresponding papers of F. Luca, L. Hajdu, R. Tijdeman, V. S. Dimitrov and E. W. Howe, Zs. Ádám and the references given there). In the presentation we extend a result of Hajdu and Tijdeman concerning the smallest number which cannot be obtained as a sum of less than k power products of fixed primes.

Put $A'_{\pm} = A' \cup (-A')$. Define the function $F(k)$ $(k \in \mathbb{N})$ to be the smallest natural number which cannot be represented as the sum of less than k terms from A', and let $F_{\pm}(k)$ be the function defined similarly, except that A' is replaced by $A'_{\pm}$.

Nathanson asked for the growth properties of $F_{\pm}(k)$, in the particular case when the elements a_i $(i = 1, \ldots, l)$ of A are primes. Hajdu and Tijdeman proved several related theorems, both for $F(k)$ and $F_{\pm}(k)$. More precisely , they proved that for all $k > 1$

$$k^{C_0^* k} < F(k) < C_1^*(kl)^{(1+\varepsilon^*)kl} \text{ and } k^{C_0^* k} < F_{\pm}(k) < \exp((kl)^{C_2^*})$$

hold, where C_0^* and C_2^* are positive absolute constants, $\varepsilon^* > 0$ is arbitrary, and C_1^* is a positive constant depending only on ε^*.

In the presentation we consider the general case, where the elements a_i $(i = 1, \ldots, l)$ of A are arbitrary positive integers. We note that it seems to be more natural to consider the problem under this condition mainly because since a part of the argument goes modulo m (with some appropriate m), the extra assumption that the numbers a_i $(i = 1, \ldots, l)$ should be primes is irrelevant at many points. To prove our results, among other things we need to extend classical results of Tijdeman concerning the gaps in A' where the a_i are primes, to the case of arbitrary positive integers a_i $(i = 1, \ldots, l)$.