About the existence of the generalized Gauss composition of means

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Let $I \subset \mathbb{R}$ be a non-void open interval. Let $M_i : I^2 \to I (i = 1, 2)$ be means on I and $a, b \in I$. Consider the sequences (a_n) and (b_n) defined by the Gauss iteration in the following way:

$$a_1 := a, \qquad\qquad b_1 := b,$$
$$a_{n+1} := M_1(a_n, b_n), \quad b_{n+1} := M_2(a_n, b_n) \quad (n \in \mathbb{N}).$$

If exist limits $\lim_{n\to\infty} a_n$, $\lim_{n\to\infty} b_n$ and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n,$$

than this common limit is called Gauss composition of the means M_1 and M_2 for the numbers a and b, and denoted by $M_1 \otimes M_2(a, b)$.

It is known, if M_1, M_2 are strict means on I, then $M_1 \otimes M_2(a, b)$ exist for every $a, b \in I$.

We generalised this result. We show, if M_1, M_2 (not necessarily continuous) means may be restricted by strict means, then exists they Gauss composition. Also show, that the continuousity of restrictive means is necessary.