On the class group of a cyclic field of odd prime power degree

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(Joint work with Cornelius Greither.)

Let p be an odd prime and $K/\mathbb{Q}$ be a Galois extension of degree $\ell = p^k$ whose Galois group $G = \mathrm{Gal}(K/\mathbb{Q})$ is cyclic. Let cl_K be the ideal class group of K and $h_K = |\,\mathrm{cl}_K\,|$ be the class number of K.

Let $p_1, \ldots, p_s$ be the primes which ramify in $K/\mathbb{Q}$, let e_j be the ramification index of p_j and g_j be the number of prime ideals of K dividing p_j. We assume that $s > 1$ and that the primes $p_1, \ldots, p_s$ are ordered in such a way that $\ell = e_1 \geq e_2 \geq \cdots \geq e_s \geq p$.

Let C_K be the Sinnott group of circular units of K, which is a subgroup of the group E_K of all units of K of finite index defined by explicit generators. Sinnott's index formula for our field K gives that the index $[E_K : C_K] = 2^{\ell-1} \cdot h_K \cdot e_2^{-1}$.

The aim of this talk is to show that, if $s > 2$, we can enlarge the Sinnott group C_K by other explicit generators to a subgroup $\overline{C}_K$ of E_K having smaller index $[E_K : \overline{C}_K] = 2^{\ell-1} \cdot h_K \cdot p^n \cdot \prod_{j=1}^{s} e_j^{-g_j}$, where $n = \sum_{i=1}^{k} \max\{g_j \mid e_j \geq p^i\}$. This formula gives that h_K is divisible by $p^{-n} \cdot \prod_{j=1}^{s} e_j^{g_j}$, which is stronger than the usual divisibility result obtained by genus theory if and only if there are at least two ramified primes p_j having $g_j > 1$. Moreover, assuming that p does not ramify in $K/\mathbb{Q}$, by a modification of Thaine-Rubin machinery we can show that if $\alpha \in \mathbb{Z}[G]$ annihilates the p-Sylow subgroup of the quotient $E_K/\overline{C}_K$ then $(1 - \sigma^{\ell/p}) \cdot \alpha$ annihilates the p-Sylow subgroup of the class group cl_K, where σ is a generator of the Galois group G.