Spectra of quadratic Pisot units as cut-and-project sets

Z. Masáková, K. Pastirčáková, E. Pelantová

The spectrum of a real number $\beta > 1$ is the set of $p(\beta)$ where p ranges over all polynomials with coefficients restricted to a finite set of consecutive integers, in particular,

$$\begin{aligned} X^r(\beta) &= \Big\{ \sum_{j=0}^{n} a_j \beta^j : n \in \mathbb{N},\, a_j \in \mathcal{A} = \{0, 1, \ldots, r\} \Big\} \\ &= \{0 = x_0 < x_1 < x_2 < x_3 < \cdots\} . \end{aligned} \tag{1}$$

The study of such sets for $\beta \in (1, 2)$ was initiated by Erdős et al. [1] and since then, many authors have contributed to the description of $X^r(\beta)$, especially in case that β is a Pisot number. A general result by Feng and Wen [2] states that for a Pisot number β and $r+1 > \beta$, the sequence of distances $x_{n+1} - x_n$ in $X^r(\beta)$ can be generated by a substitution. The alphabet of the substitution grows rapidly with r. However, neither the explicit prescription for the substitution, nor the values of distances and their frequencies are known in general. The only case of base β, for which the minimal distance in $X^r(\beta)$ is known for any r is when β is a quadratic Pisot unit [3]. For the same class of β, we show that recasting of the spectra in the frame of the cut-and-project scheme may bring new insight into the problem. We determine the values of all distances between consecutive points and their corresponding frequencies. We also show that shifting the set $\mathcal{A}$ of digits so that it contains at least one negative element, or considering negative base $-\beta$ instead of β, the generalized spectrum coincides with a cut-and-project sequence. As a consequence, we can show that the spectrum can be generated by a substitution over an alphabet at most five letters.

References

[1] P. Erdős, I. Joó, V. Komornik, *Characterization of the unique expansions* $1 = \sum_{i=1}^{\infty} q^{-n_i}$ *and related problems*, Bull. Soc. Math. France 118 (3) (1990), 377–390.

[2] D.-J. Feng, Z.-Y. Wen, *A property of Pisot numbers*, J. Number Theory, 97 (2) (2002), 305–316.

[3] T. Komatsu, *An approximation property of quadratic irrationals*, Bull. Soc. Math. France, 130 (1) (2002), 35–48.