**Distribution functions of sequences**

*Oto Strauch*

In this lecture we present two applications of distribution functions:
**Three dimensional Copula.** Applying Weyl's limit relation we compute

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}F(\gamma_q(n),\gamma_q(n+1),\gamma_q(n+2))$$
$$=\int_0^1\int_0^1\int_0^1 F(x,y,z)\mathrm{d}_x\mathrm{d}_y\mathrm{d}_x g(x,y,z)=\frac{q^4-3q^3+3q^2+2q+2}{4q^4+4q^3+4q^2},$$

where $\gamma_q(n)$ is the van der Corput sequence in base $q$, $g(x,y,z)$ is an asymptotic distribution function of $(\gamma_q(n),\gamma_q(n+1),\gamma_q(n+2))$, and $F(x,y,z)=xyz$. Here the distribution function $g(x,y,z)$ is a new copula, cf. [1].
**Two-dimensional Benford's law.** Let $x_n>0$, $y_n>0$, $n=1,2,\dots$ and $F_N(x,y)=\#\{n\le N;\{\log_b x_n\}<x \text{ and } \{\log_b y_n\}<y\}/N$ and express the integers $K_1,K_2$ in base $b$ representation as $K_1=k_1^{(1)}k_2^{(1)}\dots k_{r_1}^{(1)}$, $K_2=k_1^{(2)}k_2^{(2)}\dots k_{r_2}^{(2)}$. Denote $u_1=\log_b\left(\frac{K_1}{b^{r_1-1}}\right)$, $u_2=\log_b\left(\frac{K_1+1}{b^{r_1-1}}\right)$, $v_1=\log_b\left(\frac{K_2}{b^{r_2-1}}\right)$, $v_2=\log_b\left(\frac{K_2+1}{b^{r_2-1}}\right)$. Then we have

$$\lim_{k\to\infty}\frac{\#\{x_n \text{ has the first } r_1 \text{ digits } =K_1 \text{ and } y_n \text{ has the first } r_2 \text{ digits } =K_2\}}{N_k}$$
$$=g(u_2,v_2)+g(u_1,v_1)-g(u_2,v_1)-g(u_1,v_2),\ \text{if } \lim_{k\to\infty}F_{N_k}(x,y)=g(x,y).$$

Thus to solve the problem of digits of $(x_n,y_n)$ we need full description of the set of all distribution functions of $(\{\log_b x_n\},\{\log_b y_n\})$, see [2].

# References

[1] R.B. Nelsen. *An Introduction to Copulas. Properties and Applications*, Lecture Notes in Statistics **139** Springer-Verlag, New York, 1999.

[2] O. Strauch, Š. Porubský. *Distribution of sequences: A Sampler*, http://www.boku.ac.at/MATH/udt/