On the properties of negative base number systems associated to confluent Pisot numbers

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(Joint work with Z. Masáková and D. Dombek.)

In positive base number systems many properties are specific for the class confluent Pisot bases, i.e. zeros of $x^k - mx^{k-1} - mx^{k-2} - \cdots - mx - n$, where $k \geq 1$, $m \geq n \geq 1$. The main aspect is that any integer combinations of non-negative powers of the base with coefficients in $\{0, 1, \ldots, \lceil\beta\rceil - 1\}$ is a β-integer, although a sequence of coefficients may be forbidden in the corresponding number system, in other words

$$X(\beta) := \Big\{ \sum_{i=0}^{n} a_i \beta^i \;\mid\; n \in \mathbb{N}_0,\ a_i \in \{0, 1, \ldots, \lceil\beta\rceil - 1\} \Big\} = \mathbb{Z}_\beta. \qquad (1)$$

Confluent Pisot bases are also among the only cases where an explicit prescription for the substitution generating the spaces in $X(\beta)$ has been provided. The question of description of $X(\beta)$ is a special case of the problem about spectra of real numbers introduced by Erdős et al. We concentrate on the analogy of (1) in negative base number systems introduced by Ito and Sadahiro. We show that any integer combinations of non-negative powers of the base with coefficients in $\{0, 1, \ldots, \lfloor\beta\rfloor\}$ is a $(-\beta)$-integer, i.e.

$$X(-\beta) := \Big\{ \sum_{i=0}^{n} a_i \beta^i \;\mid\; n \in \mathbb{N}_0,\ a_i \in \{0, 1, \ldots, \lfloor\beta\rfloor\} \Big\} = \mathbb{Z}_{-\beta},$$

if and only if β is a zero of the above polynomial satisfying $m = n$ when k is even. It turns out that these are also precisely the bases, for which the infinite word $u_{-\beta}$ coding $(-\beta)$-integers has the same language as that of u_β. This fact implies some interesting properties of the corresponding system, e.g. that the language of $u_{-\beta}$ is closed under mirror image. For confluent Pisot bases, numbers with finite β-expansions form a subring of real numbers. On the other hand, for these bases (except the quadratic case) even the number $\lfloor\beta\rfloor + 1$ has no finite $(-\beta)$-representation over the alphabet $\{0, 1, \ldots, \lfloor\beta\rfloor\}$, hence the analogy does not hold. Also, as a consequence of our result, one can describe the structure of $X(-\beta)$.