Distribution functions of sequences

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In this lecture we present three applications of distribution functions of sequences.

1. The sequence $\xi(3/2)^n$ mod 1. Every distribution function $g(x)$ of $\xi(3/2)^n$ mod 1 satisfies

$$\begin{aligned} & g\Big(\frac{x}{2}\Big) + g\Big(\frac{x+1}{2}\Big) - g\Big(\frac{1}{2}\Big) \\ & \qquad = g\Big(\frac{x}{3}\Big) + g\Big(\frac{x+1}{3}\Big) + g\Big(\frac{x+2}{3}\Big) - g\Big(\frac{1}{3}\Big) - g\Big(\frac{2}{3}\Big), \end{aligned} \tag{1}$$

for $x \in [0,1]$. The following solution $g(x)$ of (1)

$$g(x) = \begin{cases} 0 & \text{for } x \in [0, 1/6], \\ 2x - 1/3 & \text{for } x \in [1/6, 3/12], \\ 4x - 5/6 & \text{for } x \in [3/12, 5/18], \\ 2x - 5/18 & \text{for } x \in [5/18, 2/6], \\ 7/18 & \text{for } x \in [2/6, 8/18], \\ x - 1/18 & \text{for } x \in [8/18, 3/6], \\ 8/18 & \text{for } x \in [3/6, 7/9], \\ 2x - 20/18 & \text{for } x \in [7/9, 5/6], \\ 4x - 50/18 & \text{for } x \in [5/6, 11/12], \\ 2x - 17/18 & \text{for } x \in [11/12, 17/18], \\ x & \text{for } x \in [17/18, 1] \end{cases}$$

satisfies Mahler's conjecture in the following sense: K. Mahler (1968) conjectured that there exists no $\xi \in \mathbb{R}^+$ such that $0 \le \{\xi(3/2)^n\} < 1/2$ for every $n = 0, 1, 2, \ldots$ Mahler's conjecture follows from the conjecture: Let $g(x)$ be a distribution function satisfying (1). Then $g(x)$ is different of $g(x) = 1$ for $x \in (1/2, 1)$.

2. The first digit problem. Let $\lim_{i\to\infty}\{\log_q(N_i)\} = w$, then for integer sequence $n = 1, 2, \dots$ we have

$$\lim_{i\to\infty} \frac{\#\{n \le N_i \text{ ; first } s \text{ digits of } n \text{ are } k_1k_2\dots k_s\}}{N_i}$$
$$= g_w\big(\log_q k_1.k_2k_3\dots(k_s+1)\big) - g_w\big(\log_q k_1.k_2k_3\dots k_s\big)$$

where

$$g_w(x) = \frac{1}{q^w}\frac{q^x-1}{q-1} + \frac{q^{\min(x,w)}-1}{q^w}$$

is a distribution function of the sequence $\log_q n \bmod 1$.

3. Four-dimensional Copula. Applying Weyl's limit relation we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1} F\big(\gamma_q(n), \gamma_q(n+1), \gamma_q(n+2), \gamma_q(n+3)\big)$$
$$= \int_0^1\int_0^1\int_0^1\int_0^1 F(x,y,z,u)\,\mathrm{d}_x\mathrm{d}_y\mathrm{d}_z\mathrm{d}_u g(x,y,z,u)$$
$$= \frac{1}{2} + \frac{3}{q} - \frac{6}{q^2}, \tag{2}$$

where

- $\gamma_q(n)$ is the van der Corput sequence in base q,
- $g(x,y,z,u)$ is an asymptotic distribution function of

$$\big(\gamma_q(n), \gamma_q(n+1), \gamma_q(n+2), \gamma_q(n+3)\big),$$

- and $F(x,y,z,u) = \max(x,y,z,u)$.

Here the distribution function $g(x,y,z,u)$ is a new copula.

Comments. Result 1. is one of the first nontrivial applications of the distribution function theory. Result 2. is a unique solution of a problem that the sequence $n = 1, 2, 3, \dots$ does not satisfy Benford's law. In Result 3. a referee described a general method for computing integral of the type (2), but 1. and 2. are given a basis for individual study of $g(x,y,z,u)$.