Diophantine problems and arithmetic progressions

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In this talk we report on recent research related to three Diophantine problems. First we consider a special case of the so-called Erdős-Graham problem. Erdős and Graham asked if the Diophantine equation

$$\prod_{i=1}^{r} f(x_i, k_i, 1) = y^2$$

has, for fixed $r \geq 1$ and $\{k_1, k_2, \ldots, k_r\}$ with $k_i \geq 4$ for $i = 1, 2, \ldots, r$, at most finitely many solutions in positive integers $(x_1, x_2, \ldots, x_r, y)$ with $x_i + k_i \leq x_{i+1}$ for $1 \leq i \leq r - 1$. Skałba provided a bound for the smallest solution and estimated the number of solutions below a given bound. Ulas answered the above question of Erdős and Graham in the negative when either $r = k_i = 4$, or $r \geq 6$ and $k_i = 4$. Tengely proved that the only solution $(x, y) \in \mathbb{N}^2$ of

$$x(x+1)(x+2)(x+3)(x+k)(x+k+1)(x+k+2)(x+k+3) = y^2, \quad (1)$$

with $4 \leq k \leq 10^6$ is

$$(x, y) = (33, 3361826160)$$

with $k = 1647$. In this talk we provide more precise answer to this problem.

Zhang and Cai deal with the equations

$$(x-1)x(x+1)(y-1)y(y+1) = (z-1)z(z+1),$$
$$(x-b)x(x+b)(y-b)y(y+b) = z^2,$$

where b is a positive even number. In case of the first equation they prove that there exist infinitely many non-trivial positive integer solutions. In case of the second equation they obtain similar result. They also pose two questions related to the above equations.

Question 1. Are all the nontrivial positive integer solutions of $(x-1)x(x+1)(y-1)y(y+1) = (z-1)z(z+1)$ with $x \leq y$ given by $(F_{2n-1}, F_{2n+1}, F_{2n}^2)$, $n \geq 1$?

Question 2. Are there infinitely many nontrivial positive integer solutions of $(x-b)x(x+b)(y-b)y(y+b) = z^2$ if $b \geq 3$ odd?

We provide some partial results related to the above questions.