Upper and lower densities – Part I

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We present an axiomatic theory of upper and lower densities on the integers that relies on a package of five axioms a real-valued set function $\mu^\star$ on the power set of $\mathbb{Z}$ is required to satisfy: monotonicity, subadditivity, (-1)-homogeneity, and translational invariance, together with the normalization condition $\mu^\star(\mathbb{Z}) = 1$. In particular, we will discuss and prove the mutual independence of these axioms, and provide a number of examples for which they are all satisfied (examples include the upper asymptotic, upper Banach, upper logarithmic, upper Buck, and upper analytic densities).