# Primitive solutions of Diophantine equations involving squares and fifth powers

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We present some result concerning solvability in integers of the Diophantine equations of the form $T^2 = G(\overline{X})$, $\overline{X} = (X_1, \ldots, X_m)$, where $m = 3$ or $m = 4$ and $G$ is a specific homogenous quintic form. First, we prove that if $F(x, y, z) = x^2 + y^2 + az^2 + bxy + cyz + dxz \in \mathbb{Z}[x, y, z]$ and $(b - 2, 4a - d^2, d) \neq (0, 0, 0)$, then for all $n \in \mathbb{Z} \setminus \{0\}$ the Diophantine equation $t^2 = nxyzF(x, y, z)$ has a solution in polynomials $x, y, z, t$ with integer coefficients, with no polynomial common factor of positive degree. In case $a = d = 0, b = 2$ we prove that there are infinitely many primitive integer solutions of the Diophantine equation under consideration. As an application of our result we prove that for each $n \in \mathbb{Q} \setminus \{0\}$ the Diophantine equation

$$T^2 = n(X_1^5 + X_2^5 + X_3^5 + X_4^5)$$

has a solution in co-prime (non-homogenous) polynomials in two variables with integer coefficients. We also present a method which sometimes allows us to prove the existence of primitive integer solutions of more general quintic Diophantine equation of the form $T^2 = aX_1^5 + bX_2^5 + cX_3^5 + dX_4^5$, where $a, b, c, d \in \mathbb{Z}$. In particular, we prove that for each $m, n \in \mathbb{Z} \setminus \{0\}$, the Diophantine equation

$$T^2 = m(X_1^5 - X_2^5) + n^2(X_3^5 - X_4^5)$$

has a solution in polynomials which are co-prime over $\mathbb{Z}[t]$. Moreover, we show how a modification of the presented method can be used in order to prove that for each $n \in \mathbb{Q} \setminus \{0\}$, the Diophantine equation

$$t^2 = n(X_1^5 + X_2^5 - 2X_3^5)$$

has a solution in polynomials which are co-prime over $\mathbb{Z}[t]$.