On the frequency of multiplicative properties in diophantine approximations of almost all real numbers

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Let E be a subset of the natural numbers and let $p_n(x)/q_n(x)$ be the n-th convergent of x related to its regular continued fraction expansion. What can be said about the set $E(x) := \{n \in E\,;\, q_n(x) \in E\}$? In the classical metric theory of continued fractions, P. Erdös (JNT 1970) proved that $\#E(x)$ is infinite for almost all x if and only if $\sum_{k \in E} \varphi(k)/k^2 = \infty$ where $\varphi(\cdot)$ is the Euler arithmetic function. A result which is closed to the Duffin-Schaeffer conjecture (that says for $\varepsilon : E \to (0, \infty)$ given, the inequality

$$|x - \frac{p}{q}| \leq \frac{\varepsilon(q)}{q}$$

holds for infinitely many p and q coprime with $q \in E$ if and only if

$$\sum_{q \in E} \varepsilon(q)\varphi(q)/q = \infty\,.$$

In this talk we first survey recent results about this conjecture and then pay more attention to sets E for which $E(x)$ has a fixed asymptotic density $\delta(E)$ for almost all x. This is precisely the case for sets E which are Buck measurable. We extend such a result to a wider class of sets E satisfying interesting multiplicative structures as, for example, the class of k-free integers. Extensions to other continued fractions in one or higher dimensions should be also discussed.