Effective results for division points on curves in $\mathbb{G}_m^2$

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Let $A := \mathbb{Z}[z_1, \ldots, z_r]$ be a finitely generated domain over $\mathbb{Z}$, and let K denote its quotient field, and denote by K^* the multiplicative group of non-zero elements of K. Let Γ be a finitely generated subgroup of K^*, and let $\overline{\Gamma}$ denote the division group of Γ. Let $F(X,Y) \in A[X,Y]$ be a polynomial. In 1960 S. Lang proved that the equation

$$F(x,y) = 0 \quad \text{in } x, y \in \Gamma \tag{1}$$

has only finitely many solutions, provided F is not divisible by any polynomial of the form

$$X^m Y^n - \alpha \quad \text{or} \quad X^m - \alpha Y^n \tag{2}$$

for any non-negative integers m, n, not both zero, and any $\alpha \in \overline{K}^*$. The conditions imposed in Lang's theorem, i.e., that Γ be finitely generated and F not be divisible by any polynomial of type (2), are essentially necessary. Lang's proof of this result is ineffective. Lang also conjectured that the above equations has finitely many solutions in $x, y \in \overline{\Gamma}$ under the same condition (2). In 1974 Liardet proved this conjecture of Lang, however, the proof of Liardet is also ineffective.

An effective version of Liardet's Theorem in the number filed case is due to Bérczes, Evertse, Győry and Pontreau (2009), however, in the general case no effective result has been proved.

In the talk an effective version of the result of Liardet will be presented in the most general case. Our result is not only effective, but also quantitative in the sense that an upper bound for the size of the solutions $x, y \in \overline{\Gamma}$ is provided. This result implies that the solutions of the equation under investigation can be determined in principle.

In the proofs we combine effective finiteness results for these types of equations over number fields and over function fields, along with a specialization method developed by Győry in the 1980's and refined recently by Evertse and Győry.