On the equation $U_n = 2^a + 3^b + 5^c$

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In the talk, first we propose a conjecture, similar to Skolem's conjecture, on a Hasse-type principle for exponential Diophantine equations. Namely, consider the equation

$$a_1 b_{11}^{\alpha_{11}} \cdots b_{1l}^{\alpha_{1l}} + \ldots + a_k b_{k1}^{\alpha_{k1}} \cdots b_{kl}^{\alpha_{kl}} = c$$

in non-negative integers $\alpha_{11}, \ldots, \alpha_{1l}, \ldots, \alpha_{k1}, \ldots, \alpha_{kl}$, where a_i, b_{ij}, are non-zero integers for every $i = 1, \ldots, k$ and $j = 1, \ldots, l$, and c is an integer. Our conjecture is that if the equation above has no solutions, then there exists an integer $m \geq 2$ such that the congruence

$$a_1 b_{11}^{\alpha_{11}} \cdots b_{1l}^{\alpha_{1l}} + \ldots + a_k b_{k1}^{\alpha_{k1}} \cdots b_{kl}^{\alpha_{kl}} \equiv c \pmod{m}$$

has no solutions in non-negative integers α_{ij}, $i = 1, \ldots, k$, $j = 1, \ldots, l$.

In the talk we present a result showing that in a sense, the conjecture is valid for "almost all" equations. Further, based upon the conjecture we propose a general method for the solution of exponential Diophantine equations, relying on a generalization of a result of Erdős, Pomerance and Schmutz concerning Carmichael's λ function.

Finally, we illustrate that our method works not only in $\mathbb{Z}$, but also in the ring of integers of $\mathbb{Q}(\alpha)$ (where α is a real algebraic number) by generalizing a result of D. Marques and A. Togbé and solving a problem of F. Luca and S. G. Sanchez. Let $U_n = A \cdot U_{n-1} + B \cdot U_{n-2}$ $(n \geq 2)$ with $A, B \in \mathbb{Z}$ and initial terms $U_0, U_1 \in \mathbb{Z}$ be a binary sequence. If a, b, c are non-negative integers, then we give all solutions of the equations

$$U_n = 2^a + 3^b,$$
$$U_n = 2^a + 3^b + 5^c,$$

in the case when $(A, B, U_0, U_1) = (1,1,0,1)$, $(1,1,2,1)$, $(2,1,0,1)$, $(2,1,2,2)$.