Bounds for exponential sums combining Van der Corput's and Huxley's method

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The classic Van der Corput's method to estimate exponential sums consists of combining Poisson's formula followed by the asymptotic evaluation of exponential integrals ("A-step"), and a skillful application of Cauchy's inequality ("B-step"). In its simplest form it tells us what follows: Let be given throughout two real parameters $M \geq 1$ and $T > 0$ with $|\log T| \asymp \log M$, and a real function F on an interval I of length M, satisfying

$$F^{(j)} \asymp M^{-j} T \,, \tag{1}$$

with $j = 2$. Then it follows that, for any interval $I^* \subseteq I$,

$$E_{F,I^*} := \sum_{n \in I^*} e^{2\pi i F(n)} \ll T^{1/2} + M T^{-1/2} \,.$$

Applying *Van der Corput's differencing lemma* (also known as Weyl's B-step), it follows that (1) holding true for $j = 3$ implies that

$$E_{F,I^*} \ll M^{1/2} T^{1/6} + M T^{-1/6} \,.$$

Of course, the application of the B-step can be iterated.

More recently, M. Huxley and others developed an entirely new approach called the *Discrete Hardy-Littlewood method.* For the single exponential sum, its sharpest result says that

$$E_{F,I^*} \ll M^{1/2} T^{32/205+\epsilon} \,,$$

provided that (1) is true for $j = 2, 3, 4$ - however, under the restriction that the parameters satisfy

$$T^{141/328+\epsilon} \ll M \ll T^{181/328} \,. \tag{2}$$

In this talk it is described how both types of results can be combined to obtain sharp bounds for the single exponential sum. Under the conditions just stated, apart from (2), it can be proved that

$$M^{-\epsilon} E_{F,I^*} \ll M^{\frac{1}{2}} \, T^{\frac{32}{205}} + T^{\frac{751}{1968}} + M^{\frac{871}{1086}} + M \, T^{-1/2} \,.$$

Applying the differencing lemma once, one obtains

$$M^{-\epsilon} E_{F,I^*} \ll M^{\frac{679}{948}}\,T^{\frac{16}{237}} + M^{\frac{1}{2}}\,T^{\frac{751}{5438}} + M^{\frac{1957}{2172}} + M\,T^{-1/4}\,,$$

under the condition that (1) is true for $j = 3, 4, 5$. Again, the B-step can be iterated.

Finally, we mention a few applications; see also [3].

References

[1] W.G. Nowak, Higher order derivative tests for exponential sums incorporating the Discrete Hardy-Littlewood method. Acta Math. Hungarica **134**/1 (2012), 12-28.

[2] W.G. Nowak, Higher order derivative tests for exponential sums incorporating the Discrete Hardy-Littlewood method, II. In preparation.

[3] W.G. Nowak, A problem considered by Friedlander and Iwaniec and the Discrete Hardy-Littlewood method. Math. Slovaca, to appear.